COMMENTS ON THE COLLEGE AND CAREER READINESS STANDARDS FOR MATHEMATICS

Submitted by the U. S. Coalition for World Class Math
www.usworldclassmath.org

I. General comments:

The *College and Career Readiness Standards for Mathematics* Draft for Review and Comment was released on September 21, 2009 as part of the Common Core State Standards Initiative. The following review and comments are provided in response to the draft.

The mathematics standards in draft form, while yet needing some work, are substantially clear and well written. Each of the topics is well defined, and most mathematical definitions are adequately precise. We applaud the emphasis on mastery, which we would like to see carried through to the grade-by-grade standards. If this work is to go forward, improvements should be made to significantly enhance the quality of the standards, core concepts, core skills, description text, and examples.

If these standards are to serve as the forerunner of future K-12 grade-by-grade objectives and standards, we believe more clarity is needed as delineated below. Our comments are intended to prevent, as much as possible, the use or abuse of this document outside the intended purpose. We encourage you to develop additional precautions to limit misinterpretation and misuse of the Standards. Examples should be focused on the important mathematics, rather than emphasizing estimation and other techniques that give students (and school administrators) the false impression that using "guess and check" and making visual approximations using graphing calculators are essential mathematical skills.

The varying level of detail in information and core skills appears to give greater priority to those standards and core skills with the greater detail. Whether intended or not, the absence or presence of detail establishes priorities. As a result, some standards that will contribute little to college readiness are given equal status with standards that are essential foundations. For example, many mathematicians believe that statistics, probability and modeling are, for the most part, non-essential for college readiness. Furthermore we believe that college math placement tests afford these topics less emphasis than is evidenced in the current draft. Given the extensive discussion of these topics in the draft standards document, however, a well meaning elementary school principal might require every teacher to devote nearly one-fourth of their math instructional time to probability and statistics. These subjects are taught quite well in colleges, and students will gain the proficiency in these topics as needed. In place of these topics, we suggest including the following skills and concepts that are typically taught in Algebra II and pre-calculus courses, and which are not reflected in the current standards:

Solutions of systems of linear equations with two or more variables using determinants
Solutions of systems of quadratic equations
Exponential equations
Logarithms
Solution of polynomial equations (Taylor's remainder theorem; synthetic division)
Binomial theorem
Permutations and combinations
Trigonometric functions
Trigonometric identities
Analytic geometry (directed distance, distance formula, midpoint formula, translation of axes, distance from point to a line)
Parametric and polar equations
Conic sections (equations for parabola, circle, ellipse, hyperbola)
Complex numbers
Solving equations with complex numbers; De Moivre's Theorem

II. Need for Higher Level of Standards

We are concerned about the following statement, which appears on p. 3 of the standards document:

> "Students reaching these levels will be prepared for non-remedial college mathematics courses and will be prepared for training programs for career-level jobs; however, the College and Career Readiness Standards for Mathematics should not be construed as grade twelve exit standards. Students interested in STEM fields, and those who wish to go beyond for other reasons, will need to reach these standards before their senior year in order to have time to include additional mathematics. A number of pathways for advanced learning are possible and may be integrated throughout the high school experience and beyond."

Obviously, students who are "interested in STEM fields" need appropriate mathematics standards just as much as other students do. It is not enough to simply state that the Common Core standards are inadequate for these students; appropriate standards must be developed for them so that teachers, school administrators, and textbook publishers can develop appropriate courses of instruction for STEM-intending students.

We strongly suggest that your group adopt the methodology Achieve, Inc. used when creating their ADP Benchmarks: They developed standards appropriate for all students – including those interested in STEM fields – and used an asterisk to identify "content that is recommended for all students, but is required for those students who plan to take calculus in college, a requisite for mathematics and many mathematics-intensive majors."

III. Specific Standards

As mentioned above, we believe that the standards for statistics, probability and modeling are largely unnecessary. Thus, we do not discuss these standards here, and we advise that these standards be eliminated (or greatly reduced in scope) and replaced with more advanced topics in mathematics. Our comments on the remaining standards follow.

A. Number

While the draft covers most of the critical content and is coherent, well organized, and clearly written, the standards are not explicit enough in addressing the arithmetic of rational numbers. The document states that "Procedural fluency in operations with real numbers and strategic competence in approximation are grounded in an understanding of place value. The rules of arithmetic govern operations on numbers and extend to operations in algebra." We agree; however, procedural fluency is impossible without automatic recall of single digit addition and multiplication facts. Automatic recall of basic facts needs to be included as a component of procedural fluency. Furthermore, estimation and approximation are addressed too extensively in comparison to the treatment of accuracy and finding exact values. The exaggerated emphasis on estimation and approximation establishes an inappropriate focus on these less-important topics.

Also, there is a reference to National Research Council's "Adding it Up". We are disappointed by the weak support of standard algorithms in "Adding it Up," although it does recommend using algorithms that are "general and reasonably efficient" and suggests that students use "adaptive reasoning to analyze and compare algorithms". We believe that educational time can be better spent by teaching students how to use the algorithms than by having students compare them. We also advise that "flexibility" be deleted from the Core Skill in Numbers; accuracy and efficiency are critical, whereas flexibility is of lesser importance and the meaning of the term is unclear.

B. Quantity

Although this is a good discussion of units and their interpretation, the term "rate" should be used throughout the discussion of quantity. Ratios should be mentioned as well. It is fine to talk about solving problems involving people-hours and social science measures such as deaths per 100,000, but this discussion should include the word "rate". Rates and ratios are often critical in solving equations involving linear relationships.

C. Expressions

In general, the explanations of algebraic expressions are clear and concise, and we are pleased to see that the core skill of completing the square is included. We note however, that pre-calculus readiness (important for students pursuing STEM fields, but also for students who may only take pre-calculus in college) involves automatic and accurate manipulation of algebraic expressions in the same way that algebra readiness involves automatic and accurate manipulation of number expressions. The consistent experience of college math departments is that STEM-intending students who have not acquired this level of algebra readiness in high school are at substantial risk for being unable to meet the demands of even "college" algebra, the course that is prerequisite to calculus.

The phrasing of Core Concept D "Rewriting expressions in equivalent forms serves a purpose in solving problems" is vague. We suggest the following: "*Rewriting expressions plays a fundamental role in solving equations and inequalities.*" It is these two activities that in turn are critical skills in pre-calculus.

We suggest also the inclusion of an additional Core Concept:

All numerical expressions simplify to the same form, namely, a single number. In contrast, there are two ways to simplify a polynomial expression: a) as a product, by factoring; and b) as a sum, by multiplying out.

D. Equations

The simplification methods for polynomials that we suggested as an additional Core Concept in Expressions play a crucial role in Equations Core Concept C. That statement: "the steps in solving an equation are guided by understanding and justified by logical reasoning" obfuscates a basic fact that all eleventh graders should know:

There is a simple algorithm for solving a polynomial equation of degree higher than one: Multiply out on each side, move everything to one side, factor that side, and set each factor to zero.

The omission of this statement from many current high school curricula leads to the curious result that large numbers of high school and college students cannot correctly solve factorable quadratic equations.

Under the heading "Equations," Core Concept D states: "Equations not solvable in one number system may have solutions in a larger number system." In the discussion this is explained by the following statement: "Some equations have no solutions in a given number system, stimulating the formation of expanded number systems (integers, rational numbers, real numbers and complex numbers)." We suggest that the explanatory statement be used in lieu of the current wording for Core Concept D, because "number systems" can be interpreted to mean different number bases.

E. Functions

Although well written, the Functions section lacks adequate specificity. Core Concept D states "Common functions occur in families where each member describes a similar type of dependence." It is never stated what specific common functions a student should be able to recognize, describe behaviors for, analyze, or use to solve problems. The text mentions linear functions and exponential functions as "important families of functions characterized by laws of growth," but no other families of functions are similarly highlighted. Core Skill 2 addresses common types of functions without specifying which ones except those for which technology is to be used to explore the effects of parameter changes. Those functions specified for using technology to explore are linear, power, quadratic, polynomial, simple rational, exponential, logarithmic, sine and cosine, absolute value and step functions. It is unclear which of these specific function types are considered to be common functions. We recommend that the Core Skills specify which functions should be addressed and delineate key characteristics of each function type. The functions should include linear, quadratic, polynomial, square and cube root, absolute value, rational, step), exponential, logarithmic, and trigonometric functions. No mention is made of inverse functions, which should be included as well.

All functions listed in NMAP's Table 1: The Major Topics of School Algebra should be included.

F. Shape

In general, these standards are well written, although we question why this category is called "Shape" when it more properly and accurately should be called "Geometry."

While the exposure to geometry in earlier grades presents formulae for the Pythagorean Theorem, areas and volumes, and their applications; high school geometry should provide the formal and rigorous derivation of them through the application of deductive reasoning and proof. Students should understand and be able to derive the formulae for areas of triangles, quadrilaterals (e.g., trapezoids and parallelograms), and other polygons. Justification of the formula for calculating the area of a circle should be part of the curriculum— some textbooks clearly explain that as the number of sides of regular polygons increases, the area approaches that of a circle. Other, simpler techniques are also available. Similarly, students should be able to derive and prove the formulae for calculating the volumes of prisms, and should understand the theorems for calculating the volumes of pyramids, cylinders, and cones. The volume of a pyramid is admittedly difficult to understand, but the proof of the theorem should be part of a standard course of geometry. Students should be able to apply Cavalieri's principle and be able to apply it to derive the formula for the volume of a cone. Surface area and volume of a sphere should be presented, and not merely given to the student as formulae to memorize. Again, there are textbooks that provide excellent presentations of this material (Moise and Downs, Jurgenson, Jacobs, and Adkins/Weeks, for example).

We respect the statement “From only a few axioms, the deductive method of Euclid generates a rich body of theorems about geometric objects, their attributes and relationships.” Too often, however, textbooks present topics in a random sequence so that, for example, students are expected to prove geometric properties using coordinate geometry before the Pythagorean Theorem has been presented and proven. The midsegment theorem for triangles is proven using coordinate geometry and then the theorem is used in synthetic geometry.

The power and significance of Euclid's axioms are most effectively conveyed when the order of presentation supports the consistent and diligent application of deductive reasoning. Theorems are built up from the most basic of concepts and from theorems that have been proven. For example, students should not be presented with the midsegment theorem for triangles until they have been learned the key theorems of parallel lines which are then used to prove the midsegment theorem.

In light of this, Core Skill Number 2 is problematic. It states “Prove theorems, test conjectures and identify logical errors. Include theorems establishing the properties in Core Skill 1 and other theorems about angles, parallel and perpendicular lines, similarity and congruence of triangles.” It is not clear what is meant by “theorems”. Any proposition is a theorem, but does the statement mean only key theorems, or general propositions in the form of problems? Also, proofs should be “deductive proofs” not inductive and based on conjectures. The Core Skill should be reworded to read “Students should be able to understand and write deductive proofs of geometric propositions involving the properties in Core Skill 1, and other propositions about angles, parallel and perpendicular lines, similarity and congruence of triangles, areas of triangles, quadrilaterals and polygons, properties of circles, and properties of prisms, cylinders, pyramids, cones and spheres.”

Finally, this standard states: “Limit angle measures to degrees.” We question why limit it only to degree? Why not introduce the concept of radian, which is used extensively in trigonometry and calculus. Geometry would be a good place to introduce this measure, particularly in discussing arc length.

G. Coordinates

We believe that the standard for "coordinates" can be combined with functions, since they are very closely related. On page 16, under Core Skill #1 (Translate fluently between lines in the coordinate plane and their equations) repeats the Point-Slope Form which was discussed on page 13 (Functions) under Core Skill #3.

The discussion on page 16 states: "Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling and proof." We agree with this, but believe that students also understand how geometric principles connect with coordinate geometry. For example, the principle that a straight line has a constant slope can be proven using similar triangles. Also, the principle that the slopes of perpendicular lines are the negative reciprocals of each other (or m1m2 = -1) can also be proven using triangle congruence theorems.

III. Examples

Example problems are a key component of a standards document: They must illuminate the level of complexity, breadth of scope, and depth of knowledge intended for each item. Both quantity and quality of example problems must be high. A large number of example problems are needed to ensure that expectations for each standards item are adequately clarified. High quality is also crucial. Dramatic improvement is needed in both areas: The current number of example problems is inadequate and many of the existing example problems are seriously flawed. In the current draft, many standards items are not represented by example problems, leaving the reader to wonder what level of complexity, breadth of scope, and depth of knowledge are intended. Another concern is that many of the example problems focus on topics that are mathematically nonessential.

A very serious shortcoming of the current standards draft is the deficiencies in the set of the example problems. It is difficult to overstate the importance of improving the quantity, quality, and relevance of the example problems.

In this section, we offer examples of flawed problems and recommendations for improvement.

A. Equations:

9. Core Skill 1. One firm offers an investment plan that pays a flat rate of 10% interest each year on the original sum invested. So each dollar grows after n years to (1 +10n/100) dollars.

Another firm offers a plan that pays 5% interest each year on the previous year's balance. So each dollar grows after n years To $(1 + 5/100)^n$ dollars.

Find, using a graphing calculator or a spreadsheet, when the two offers give roughly equal returns. Which is better in the long term?

Comment: Core Skill 1 of the Equations standard states: "Understand a problem and formulate an equation to solve it". While a student may be able to express the equation, he/she will not be able to solve it except by visual inspection using a graphing calculator which is not the goal of Core Skill 1 or what equation solving is about. Solving an equation of this type is not typically encountered in algebra 2 or even precalculus. This problem contains very little that is mathematically interesting or useful and carries a message of: "The only way you can solve this problem is through estimation techniques using a graphing calculator."

A better application of exponential equations (presumably for algebra 2 or precalculus in which students have learned how to operate with logarithms) would be to ask the student to find how many years 1000 dollars would have to be invested at 5% interest compounded annually to yield $100 of interest, assuming the bank awards interest annually only and does not award a portion of the interest for a fraction of a year. Given the formula above, the student will then need to solve the equation $1.05^n = 1{,}100$*. To solve it, the student would take the logarithm of each side and then solve for n, so that* $n = \log 1{,}100/\log 1.05$*. The result is 17.3 years. Since the bank does not prorate interest, the money would need to be invested for 18 years*

Alternatively, the problem would also be improved if students were required to create the mathematical expressions themselves, rather than being given them by the authors.

25. Core Skill 3.

If oil should ever be spilled into the Columbia River Estuary, the company responsible for the spill would be liable for monetary damages according to a formula. By Washington state law, the formula in 2009 was given by:

D = 0:508GS(A + B + C)

In this formula, D is the damage liability in dollars; G is the number of gallons spilled; S is a "vulnerability score" in the range from 1 to 5 that takes into account the wildlife characteristics of any given square kilometer of the estuary; and A, B and C are "chemical penalty scores" in the range from 1 to 5 that take into account the toxicity, harmful mechanical properties, and longevity of the material spilled. For example, kerosene has a toxicity score A = 1:4, a harmful mechanical property score B = 2:4, and a longevity score C = 1. Suppose that a company responsible for a kerosene spill in an area of lowest vulnerability is held liable for $10 million. How many gallons were spilled? How many dollars per gallon was the company charged for the spill? In general, what is a formula for the number of dollars of liability per gallon of spill? What is the maximum possible liability in dollars per gallon?

Comments: The challenge posed to students by this problem is largely due to the needlessly complex terminology and other linguistic challenges. The underlying mathematical competencies could and should be assessed with much simpler verbiage.

B. Quantity:

1. Core Concept A.

The following question was posted on an Internet gardening forum:

I am trying to figure out how many yards of soil I need for my gardens but have no idea how much a yard of soil actually is. The nursery says they deliver 6 yards in a dump truck. I realize that a yard is 3 feet. But what is a yard of soil—is that 3 feet long and 3 feet high? I'm clueless! LOL
Write a helpful response to the person who posted this question.

Comments: The difficulty of this problem arises from non-mathematical issues. Students who are not familiar with the horticultural use of the term “yard” *will be unfairly penalized. Also, competency with the underlying mathematics is improperly conflated with facility in generating prose writing.*

3. Core Concept B, Core Skill 4.

As a flooring contractor, Lupe sets floor tile for a living. She submits a bid for each new job. When preparing a bid, she measures the area of the floor to be tiled and then figures out how much material she will need. She charges the following prices for materials and labor:

_ Subflooring: $1.27 per square foot
_ Tile: $6.59 per square foot
_ Adhesive: $31.95 per job
_ Grout: $55.95 per job
_ Labor: $125 base price plus $0.79 per square foot

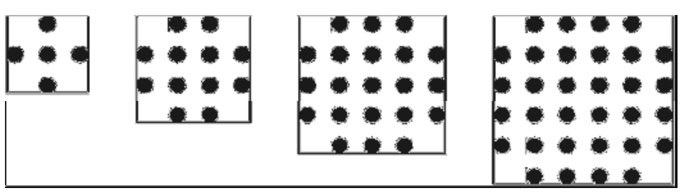
For a job tiling an area of 550 square feet, what is the amount of the bid, based on the materials listed above?

Comments: The question asks for the amount of the bid “based on the materials listed above.” “Labor” is included in the list; but it is not clear whether students are supposed to include or exclude the labor cost.

C. Expressions:

12. Core Concept D; Core Skill 1.
Use Figure 2 to explain why the following expressions are equivalent:

$(n+2)^2 -4$ and $n^2 + 4n$



Comments: This example requires students to exhibit mathematically flawed reasoning. Observing that a rule holds for first four cases does not imply that it can be generalized to all cases, and it is improper to encourage students to adopt this faulty reasoning.

Also, the difficulty of the problem arises largely because of the authors’ failure to specify what is represented by the symbol n. To do the problem, students must see that the dots that would fill an entire square are $(n+2)^2$, where n is the number of dots in each of the outermost edges of the square. They must also see that each square is missing the dot in each of the 4 corners and in so doing, can then equate the picture to the first expression. Using similar reasoning the student is to justify how the second expression describes the number of dots.

This problem presumably uses Core Concept D and Core Skill 1. Core Concept D is: "Rewriting expressions in equivalent forms serves a purpose in solving problems." Core Skill 1 is: "See structure in expressions." It could achieve these in a manner far more useful for students by asking them to show algebraically how the two expressions are equivalent, thus reinforcing the concept of binomial forms and algebraic manipulation.

As the problem is stated, it appears to be developing a "habit of mind" whose purpose is unclear. Applications of algebra are useful only to the extent that they reinforce the concepts. This problem would most likely confuse rather than reinforce.

13. Core Concept D; Core Skill 1.

Write a possible identity for Line n, and show how it fits the pattern for each line in the list given.

(a)
_ Line 1: 1 x 3 = 2 + 1
_ Line 2: 2 x 4 = 4 + 4
_ Line 3: 3 x 5 = 6 + 9
_ Line 4 4 x 6 = 8 + 16
_ Line 5: 5 x 7 = 10 + 25

(b)
_ Line 1: 1 - 1 = 2 x 0
_ Line 2: 4 - 1 = 3 x 1
_ Line 3: 9 - 1 = 4 x 2
_ Line 4: 16 - 1 = 5 x 3
_ Line 5: 25 - 1 = 6 x 4

Comments: "A reliance on general understanding or ingenuity beyond the level of the actual mathematics involved [constitutes a flaw]"

- *Foundations for Success: The National Mathematics Advisory Panel Reports of the Task 2008 Groups and Subcommittees, page 8-28*
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This problem is essentially a mathematical puzzle, and success in solving it may correlate more with innate intelligence and ingenuity than with mastery of the underlying mathematics

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10/26/09 Addendum to Comments

The comments we submitted on 10/21/09 emphasized our concern that the Common Core Standards fail to specify the optional, higher-level mathematical content necessary for college-readiness in STEM disciplines. It has now come to our attention that enrollment prerequisites for BA programs in *non-STEM* fields of many, perhaps most, state universities also require mastery of numerous Algebra II and Geometry topics that are not included in the current draft. This includes the California State University and University of California systems, the University of Texas and Texas A&M systems, University of Illinois and Illinois State University systems, Florida State and University of Florida, Ohio State University, and many others.

This omission of significant portions of essential Algebra II and Geometry content renders the Common Core Standards inadequate for students who will enter undergraduate programs in STEM or even non-STEM disciplines in much of the country. We recommended in our comments that the authors of Common Core Standards follow the methodology Achieve, Inc. used in developing the ADP Benchmarks: Create standards appropriate for *all* students by specifying essential content up to and including that needed in preparation for study in STEM fields, and identifying as optional those topics required only for students wishing to prepare for study in STEM disciplines. States should not adopt the "College-Readiness" Standards unless they adequately identify the content required for success in credit-bearing mathematics courses in their state universities. The current Common Core Standards draft falls significantly short of this requirement for many states.